Recursive Least-Squares Algorithm

Let

$$z(t) = \sum_{i=1}^{N} w_i r_i(t) \tag{1}$$

with the task of making $z(t) \approx f(t)$. The least-square solution is

$$w_i = \sum_{j=1}^{N} P_{ij} D_j \tag{2}$$

where

$$P_{ij} = \left(C^{-1}\right)_{ij} \quad \text{and} \quad D_j = \langle r_j f \rangle \tag{3}$$

with

$$C_{ij} = \langle r_i r_j \rangle \, . \tag{4}$$

Doing this recursively, we write

$$\begin{aligned} w_i(t) &= w_i(t-1) + \Delta w_i(t) \\ C_{ij}(t) &= C_{ij}(t-1) + r_i(t) r_j(t) \\ P_{ij}(t) &= P_{ij}(t-1) + \Delta P_{ij}(t) \\ D_j(t) &= D_j(t-1) + r_j(t) f(t) \, . \end{aligned} \tag{5}$$

In the following, r_i always means $r_i(t)$ and f means $f(t)$. Now,

$$\Delta P_{ij}(t) = -\frac{\left(\sum_k P_{ik}(t-1) r_k\right)\left(\sum_l r_l P_{lj}(t-1)\right)}{1 + \sum_{k,l} r_k P_{kl}(t-1) r_l} \, . \tag{6}$$

A useful identity arising from this results is

$$\sum_j P_{ij}(t) r_j = \frac{\sum_j P_{ij}(t-1) r_j}{1 + \sum_{k,l} r_k P_{kl}(t-1) r_l} \, . \tag{7}$$

We can write

$$\Delta w_i(t) = \sum_j \left(\Delta P_{ij}(t) D_j(t-1) + P_{ij}(t) r_j f\right) \tag{8}$$

and

$$\sum_j \Delta P_{ij}(t) D_j(t-1) = -\frac{\left(\sum_k P_{ik}(t-1) r_k\right)\left(\sum_{j,l} r_l P_{lj}(t-1) D_j(t-1)\right)}{1 + \sum_{k,l} r_k P_{kl}(t-1) r_l} \tag{9}$$

but

$$\sum_{j,l} r_l P_{lj}(t-1) D_j(t-1) = \sum_l w_l(t-1) r_l \, . \tag{10}$$

so

$$\sum_j \Delta P_{ij}(t) D_j(t-1) = -\frac{\left(\sum_k P_{ik}(t-1) r_k\right)\left(\sum_l w_l(t-1) r_l\right)}{1 + \sum_{k,l} r_k P_{kl}(t-1) r_l} \tag{11}$$

Using equation 7, this gives

$$\sum_j \Delta P_{ij}(t) D_j(t-1) = -\sum_j P_{ij}(t) r_j \sum_k w_k(t-1) r_k \,. \tag{12}$$

Putting equations 8 and 12 together.

$$\Delta w_i(t) = \sum_j P_{ij}(t) r_j e(t) \tag{13}$$

where

$$e(t) = f(t) - \sum_k w_k(t-1) r_k(t) \tag{14}$$

and with equation 6 giving $P_{ij}(t)$.