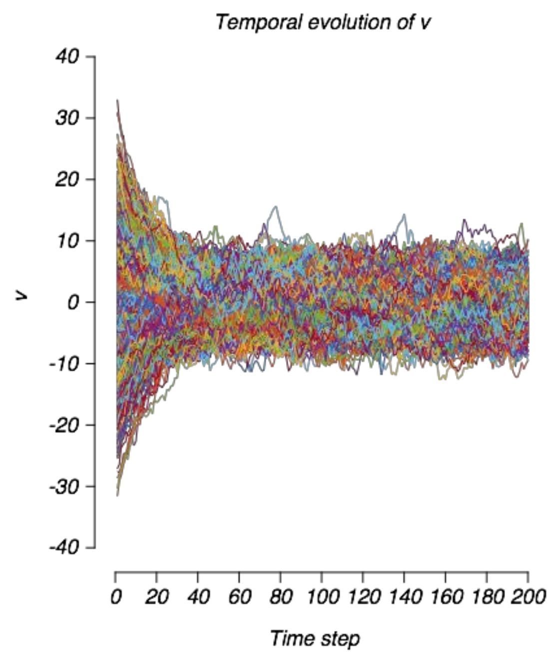
Stochastic differential equations (SDE) problem set

Consider the stochastic differential equation

(1) $$\frac{\partial v(t)}{\partial t} = -av(t) + b\eta(t)$$

where $\eta(t)$ is zero mean Gaussian white noise with unit variance. Equation (1) is a specific case of an *Ornstein-Uhlenbeck (OU) process*, and describes the evolution of the variable *v* over time.



1. Simulation algorithm. To gain intuition for how the variable *v* evolves over time in the OU process, we can approximate the solution via numerical simulation. Write a function to simulate this process using the following algorithm:

(2) $$v(i,j) = v(i,j-1) - av(i,j-1) + b\eta(i,j)$$

where *v*(*i,j*) is the voltage for trial *i* at time step *j*. *a* and *b* are parameters supplied by the user.

Simulate 1000 trials of 200 time steps with *a* = 0.05 and *b* = 1. Each new trial *i*, select *v*(*i*,1) from a Gaussian distribution with zero mean and variance equal to 100 to initialize *v*. and confirm that the temporal profile of *v* across trials qualitatively matches the figure above.

2. Simulate the "diffusion" process. First, let's study how the noise term, $b\eta(t)$, influences the evolution of *v*. Set *a* = 0 and *b* = 1 and simulate 1000 trials of 200 time steps of the OU process. Plot the mean and variance of the variable *v* across trials as a function of time. Save the results of this simulation for later.

3. Analytical description of the diffusion process. Now we want to understand the distribution *p*(*v*,*t*), which describes *v* over at each moment in time. It turns out that *p*(*v*,*t*) is a Gaussian with mean 0 and variance, $\sigma^2(t)$, that changes with time. Therefore we write:

(3) $$p(v,t) = \frac{1}{\sigma(t)\sqrt{2\pi}} e^{-\frac{v2}{2\sigma 2(t)}}$$

a) Using the chain and product rules of calculus, show

(4) $$\frac{\partial}{\partial t} p(v,t) = p(v,t)\left(\frac{v^2}{\sigma^2(t)} - 1\right)\left(\frac{1}{\sigma(t)}\right)\left(\frac{d}{dt}\sigma(t)\right)$$

Hint: use the following identities:

$x = \sigma(t)$ $\quad y = x^2$ $\quad z = \frac{v^2}{2y}$

$\frac{dx}{dt} = \frac{d}{dt}\sigma(t)$ $\quad \frac{dy}{dx} = 2x$ $\quad \frac{dz}{dy} = -\frac{v^2}{2y^2}$

b) The Fokker-Planck equation provides an alternate method for describing the evolution of the probability density. For diffusion, the Fokker-Planck equation states $\frac{\partial}{\partial t}p(v,t) = \frac{b^2}{2}\frac{\partial^2}{\partial^2 v}p(v,t)$ (notice the partial derivative on the right hand side is with respect to *v*). Assuming the p(v,t) is Gaussian as defined in equation (3), use the Fokker-Planck equation to show

(5) $\frac{\partial}{\partial t}p(v,t) = p(v,t)\left(\frac{v^2}{\sigma^2(t)} - 1\right)\left(\frac{b^2}{2\sigma^2(t)}\right)$

c) Set the results of equations (4) and (5) equal to each other and solve for $\frac{d}{dt}\sigma(t)$. With $\sigma^2(0)$ equal to the variance at *t* = 0. Use this result to show $\frac{d}{dt}\sigma^2(t) = b^2$ and $\sigma^2(t) = \sigma^2(0) + b^2 t$. Compare this analytical derivation of the variance over time to that found via numerical simulation.

d) What can we say about the variance of the diffusion process as a function of time? What is the variance of the diffusion process as $t \to \infty$?

4. Simulation of the "drift" process. Now consider the deterministic portion of the OU process. Repeat the simulation from question 2, but set *a* = 0.05 and *b* = 0. Plot the resulting mean and variance of *v* over time. How does this differ from the diffusion process of question 2? Save the results of this simulation for later.

5. Analytical description of the drift process. According to the Fokker-Planck equation for the drift process

(6) $\frac{\partial}{\partial t}p(v,t) = -\frac{\partial}{\partial v}\Big(-avp(v,t)\Big)$

a) Set equation (4) equal to (6) to show $\frac{d}{dt}\sigma(t) = -a\sigma(t)$. The solution to this differential equation is $\sigma(t) = \sigma(0)e^{-at}$. What can we say about the variance over time?

b) What is the variance for $t \to \infty$? Compare this result to that found by simulation of the drift process and the results of the diffusion process.

c) Based on the simulation and analytical results, why do you think the OU process is often called a "mean-reverting" process?

6. Simulation of the full OU process - drift and diffusion. Now simulate the OU process as in questions 2 and 4, but with *a* = 0.05 and *b* = 1. Now *both* the drift and diffusion processes influence *v* over time.

a) What do you predict the variance to be at $t \to \infty$?

b) Compare the resulting variance over time to the variance of *v* if you simply added the results from the diffusion and drift simulations alone. Are they the same? Postulate why or why not.

7. Analytical description of the full OU process. The full Fokker-Planck equation for this OU process is written

(7) $$\frac{\partial}{\partial t}p(v,t) = -\frac{\partial}{\partial v}(-avp(v,t)) + \frac{b^2}{2}\frac{\partial^2}{\partial^2 v}p(v,t)$$

Which combines the drift (6) and diffusion (5) terms from above linearly.

a) Using equation (4) and the full Fokker-Planck equation (7), find $\frac{d}{dt}\sigma(t)$.

b) Using this result, show $\frac{d}{dt}\sigma^2(t) = -2a\sigma^2(t) + b^2$.

Solving the differential equation in 6b, we find the equation for the variance as a function of time:

(8) $$\sigma^2(t) = \sigma^2(0)e^{-2at} + \frac{b^2}{2a}(1 - e^{-2at})$$

c) How does the initial variance of *v* influence the variance as *t* increases?
d) What is the variance of the OU process as $t \to \infty$?
e) How do the drift and diffusion terms (*a* and *b*) influence the p(*v*,*t*)?