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Brain Signals Localization by Alternating Projections

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We present a novel solution to the problem of localization of brain signals. The solution is sequential and iterative, and is based on minimizing the least-squares (LS) criterion by the alternating projection (AP) algorithm, well known in the context of array signal processing. Unlike existing solutions belonging to the linearly constrained minimum variance (LCMV) and to the multiple-signal classification (MUSIC) families, the algorithm is applicable even in the case of a single sample and in the case of synchronous sources. The performance of the solution is demonstrated via simulations.

Index Terms

Brain signals, EEG, MEG, least-squares, alternative projections, iterative beamformer, MUSIC, iterative-sequential weighted-MUSIC, iterative-sequential MUSIC, fully correlated sources, synchronous sources.

I. Introduction

Brain signals, obtained by an array of electric or magnetic sensors using electroencephalography (EEG) or magnetoencephalography (MEG), offer the potential to investigate the complex spatiotemporal activity of the human brain. As a result, this area of research has gained much interest in recent years [1].

Many methods have been developed for the localization of brain-signals sources, including the minimum norm [2], [3], [4], beamforming [5], [6] [7], multiple signal classification (MUSIC) [8], [9], [10] and independent component analysis (ICA) [11]. A common problem to all these methods is that they fail in the case of fully correlated sources, referred to as synchronous sources.

Though several methods aimed at coping with synchronous sources have been developed [12], [13], [14], [15], [16], [17], [18], [19], [20], they either require some a-priori information on the location of the synchronous sources, are limited to the localization of pairs of synchronous sources, or are limited in their performance.

In this paper we present a novel solution to this problem which does not have these limitations. Our starting point is the least squares (LS) estimation criterion for the problem. As is well known, this criterion yields a multidimensional nonlinear and nonconvex minimization problem, making it very challenging to avoid being trapped in undesirable local minima [21]. To overcome this challenge we resort to the alternating projection (AP) algorithm [22], well

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known in sensor array signal processing. The AP algorithm transforms the multidimensional problem to an iterative process involving only 1-dimensional maximization problems, which are computationally much simpler. Moreover, the algorithm has a very effective initialization scheme that is the key to its good global convergence.

As the AP algorithm minimizes the LS criterion, which coincides with the maximum likelihood (ML) criterion in case the noise is white and Gaussian, it enjoys the following well-known properties: (i) good performance in low signal-to-noise ratio (SNR) and in low number of samples, not failing even in the case of a *single sample*, (ii) good resolution between closely spaced source, and (iii) ability to cope with any mixture of correlated sources, including *synchronous sources*.

The rest of the paper is organized as follows. In section II we formulate the problem. Then, in section III we present the AP solution. Section IV presents the simulation results and section V the concluding remarks.

II. Problem Formulation

Consider an array composed of M EEG or MEG sensors. Assume that Q equivalent current dipole (ECD) sources, located at locations $\{\mathbf{p}_q\}_{q=1}^{Q}$, are emitting signals $\{s_q(t)\}_{q=1}^{Q}$ that are received by the array.

Under these assumptions, the $M \times 1$ vector of the received signals by the array is given by

$$\mathbf{y}(t) = \sum_{q=1}^{Q} \mathbf{l}(\mathbf{p}_q) s_q(t) + \mathbf{n}(t), \tag{1}$$

where $\mathbf{l}(\mathbf{p}_q)$ is the topography of the dipole at location $\mathbf{p}_q$ and $\mathbf{n}(t)$ is the noise. The topography $\mathbf{l}(\mathbf{p}_q)$, is given by

$$\mathbf{l}(\mathbf{p}_q) = \mathbf{L}(\mathbf{p}_q)\mathbf{q}, \tag{2}$$

where $\mathbf{L}(\mathbf{p}_q)$ is the $M \times 3$ lead-field matrix at location $\mathbf{p}_q$ and $\mathbf{q}$ denotes the 3×1 vector of the orientation of the ECD source. Depending on the problem, the orientation $\mathbf{q}$ may be known, referred to as *fixed-oriented*, or it may be unknown, referred to as *freely-oriented*.

Assuming that the array is sampled N times at $t_1, ..., t_N$, the matrix $\mathbf{Y}$ of the sampled signals can be expressed as

$$\mathbf{Y} = \mathbf{A}(\mathbf{P})\mathbf{S} + \mathbf{N}, \tag{3}$$

where $\mathbf{Y}$ is the $M \times N$ matrix of the received signals

$$\mathbf{Y} = [\mathbf{y}(t_1), ..., \mathbf{y}(t_N)], \tag{4}$$

$\mathbf{A}(\mathbf{P})$ is the $M \times Q$ matrix of the topography vectors of the Q locations (to simplify the notation, the explicit dependence on the locations $\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_Q]$ will be sometimes dropped)

$$\mathbf{A}(\mathbf{P}) = \mathbf{A} = [\mathbf{l}(\mathbf{p}_1), ..., \mathbf{l}(\mathbf{p}_Q)], \tag{5}$$

$\mathbf{S}$ is the $Q \times N$ matrix of the sources

$$\mathbf{S} = [\mathbf{s}(t_1), ..., \mathbf{s}(t_N)] = [\mathbf{s}_1^T, ..., \mathbf{s}_Q^T]^T, \tag{6}$$

with

$$\mathbf{s}(t) = [s_1(t), ..., s_Q(t)]^T, \tag{7}$$

and $\mathbf{N}$ is the $M \times N$ matrix of the noise

$$\mathbf{N} = [\mathbf{n}(t_1), ..., \mathbf{n}(t_N)]. \tag{8}$$

We further make the following assumptions regarding the emitted signals and the propagation model:

A1: The number of sources Q is *known* and obeys $Q < M$.

A2: The emitted signals are *unknown* and *arbitrarily correlated*, including the case that a subset of the sources or all of them are *synchronous*.

A3: The lead-field matrix $\mathbf{L}(\mathbf{p})$ is *known* for every location $\mathbf{p}$ (computed by the *forward model*).

A4: Every Q topography vectors $\{\mathbf{l}(\mathbf{p}_q)\}_{q=1}^{Q}$ are linearly independent, i.e., $\text{rank}\mathbf{A}(\mathbf{P}) = Q$.

We can now state the brain signals localization problem as follows: *Given the received data* $\mathbf{Y}$*, estimate the* Q *locations of the sources* $\{\mathbf{p}_q\}_{q=1}^{Q}$ and their time-course $\{\mathbf{s}_q\}_{q=1}^{Q}$.

III. The Alternating Projection Solution

We first solve the problem for the fixed-oriented dipoles and then extend it to freely-oriented dipoles.

The least squares estimation criterion for the problem, from (3), is given by

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{P},\mathbf{S}} \| \mathbf{Y} - \mathbf{A}(\mathbf{P})\mathbf{S}\|_F^2. \tag{9}$$

where F denotes the Forbenius matrix norm. To solve this minimization problem, we first eliminate the unknown signals matrix $\mathbf{S}$ by expressing it in terms of $\mathbf{P}$. To this end, we equate to zero the derivative of (9) with respect to $\mathbf{S}$, using the well known matrix differentiation rules, and solve for $\mathbf{S}$. Dropping the explicit dependence of $\mathbf{A}$ on $\mathbf{P}$, we get

$$\hat{\mathbf{S}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y}. \tag{10}$$

Now, substituting (10) into (9), yields

$$\hat{\mathbf{P}} = \arg\min_{\mathbf{P}} \| \mathbf{Y} - \mathbf{P}_{\mathbf{A}(\mathbf{P})}\mathbf{Y}\|_F^2, \tag{11}$$

where $\mathbf{P}_{\mathbf{A}}$ is the projection matrix on column span of $\mathbf{A}$

$$\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T. \tag{12}$$

Using the properties of the trace and projection operators, (11) reduces to

$$\hat{\mathbf{P}} = \arg\max_{\mathbf{P}} \text{tr}(\mathbf{P}_{\mathbf{A}(\mathbf{P})}\mathbf{C}). \tag{13}$$

where $\mathbf{C}$ is the covariance matrix given by

$$\mathbf{C} = \mathbf{Y}\mathbf{Y}^T. \tag{14}$$

This is a nonlinear and nonconvex Q-dimensional maximization problem. The AP algorithm [22] solves this problem by transforming it to a sequential and iterative process involving only 1-dimensional maximization problems.

The transformation is based on the projection-matrix decomposition formula. Let $\mathbf{B}$ and $\mathbf{D}$ be two matrices with the same number of rows, and let $\mathbf{P}_{[\mathbf{B},\mathbf{D}]}$ denote the projection-matrix onto the columns space of the augmented matrix $[\mathbf{B},\mathbf{D}]$. It is well known that

$$\mathbf{P}_{[\mathbf{B},\mathbf{D}]} = \mathbf{P}_{\mathbf{B}} + \mathbf{P}_{\mathbf{D}_{\mathbf{B}}}. \tag{15}$$

where $\mathbf{D}_{\mathbf{B}}$ is the matrix defined as

$$\mathbf{D}_{\mathbf{B}} = (\mathbf{I} - \mathbf{P}_{B})\mathbf{D}. \tag{16}$$

The AP algorithm exploits this decomposition to transform the multidimensional maximization (13) into a sequential and iterative process involving only a maximization over a single parameter at a time, with all the other parameters held fixed at their pre-estimated values. More specifically, the value of $\mathbf{p}_q$ at the $(j+1)$-th iteration is obtained by solving:

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q} \operatorname{tr}(\mathbf{P}_{[\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)}),\mathbf{l}(\mathbf{p}_q)]}\mathbf{C}). \tag{17}$$

where $\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})$ is the $M \times (Q-1)$ matrix given by

$$\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)}) = [\mathbf{l}(\hat{\mathbf{p}}_1^{(j+1)}), ..., \mathbf{l}(\hat{\mathbf{p}}_{q-1}^{(j+1)}), \mathbf{l}(\hat{\mathbf{p}}_{q+1}^{(j)}), ..., \mathbf{l}(\hat{\mathbf{p}}_Q^{(j)})]. \tag{18}$$

By the projection matrix decomposition (15), we have

$$\mathbf{P}_{[\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)}),\mathbf{l}(\mathbf{p}_q)]} = \mathbf{P}_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})} + \mathbf{P}_{\mathbf{l}(\mathbf{p}_q)_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})}}. \tag{19}$$

where $\mathbf{l}(\mathbf{p}_q)_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})}$ is the $M \times 1$ vector given by

$$\mathbf{l}(\mathbf{p}_q)_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})} = (\mathbf{I} - \mathbf{P}_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})})\mathbf{l}(\mathbf{p}_q). \tag{20}$$

Substituting (19) into (17), ignoring the contribution of the first term since it is not a function of $\mathbf{p}_q$, we get

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q} \operatorname{tr}(\mathbf{P}_{\mathbf{l}(\mathbf{p}_q)_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})}}\mathbf{C}), \tag{21}$$

which, from (12) and (20), using the properties of the trace operator, can be rewritten as

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q} \frac{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(j)}\mathbf{C}\mathbf{Q}_{(q)}^{(j)}\mathbf{l}(\mathbf{p}_q))}{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(j)}\mathbf{l}(\mathbf{p}_q)}, \tag{22}$$

where $\mathbf{Q}_{(q)}^{(j)}$ is the projection matrix that projects out all but the q-th source

$$\mathbf{Q}_{(q)}^{(j)} = (\mathbf{I} - \mathbf{P}_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(j)})}). \tag{23}$$

The initialization of the algorithm, which is the key to its good global convergence, is very straightforward. First we solve (13) for a single source:

$$\hat{\mathbf{p}}_1 = \arg\max_{\mathbf{p}_1} \frac{\mathbf{l}^T(\mathbf{p}_1)\mathbf{C}\mathbf{l}(\mathbf{p}_1))}{\mathbf{l}^T(\mathbf{p}_1)\mathbf{l}(\mathbf{p}_1)}. \tag{24}$$

Then, we add one source at a time and solve for the q-th source, $q = 2, ..., Q$, by

$$\hat{\mathbf{p}}_q = \arg\max_{\mathbf{p}_q} \frac{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(0)}\mathbf{C}\mathbf{Q}_{(q)}^{(0)}\mathbf{l}(\mathbf{p}_q))}{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(0)}\mathbf{l}(\mathbf{p}_q)}, \tag{25}$$

where $\mathbf{Q}_{(q)}^{(0)}$ is the projection matrix that projects out the previously estimated $q-1$ sources:

$$\mathbf{Q}_{(q)}^{(0)} = (\mathbf{I} - \mathbf{P}_{\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(0)})}), \tag{26}$$

with $\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(0)})$ being the $M \times (q-1)$ matrix given by

$$\mathbf{A}(\hat{\mathbf{P}}_{(q)}^{(0)}) = [\mathbf{l}(\hat{\mathbf{p}}_1), ..., \mathbf{l}(\hat{\mathbf{p}}_{q-1})]. \tag{27}$$

Once the location of the Q-th source has been estimated, the second phase of the algorithm, described by (22), is starting. The iterations continue till the marginal improvement from one iteration to the next is less than a pre-defined threshold.

Note that the algorithm climbs the peak of (13) along lines parallel to $\mathbf{p}_1, ...\mathbf{p}_Q$, with the rate of climb depending on the structure of (13) in the proximity of the peak. Since a maximization is performed at every step, the value of the maximized function cannot decrease. As a result, the algorithm is bound to converge to a local maximum which may not necessarily be the global one. Yet, the above described initialization phase assures good global convergence.

In the case of freely-oriented dipoles, it follows from (2) that the maximization problem (22) becomes:

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q, \mathbf{q}} \frac{\mathbf{q}^T \mathbf{L}^T(\mathbf{p}_q) \mathbf{Q}_{(q)}^{(j)} \mathbf{C} \mathbf{Q}_{(q)}^{(j)} \mathbf{L}(\mathbf{p}_q)) \mathbf{q}}{\mathbf{q}^T \mathbf{L}^T(\mathbf{p}_q) \mathbf{Q}_{(q)}^{(j)} \mathbf{L}(\mathbf{p}_q) \mathbf{q}}, \tag{28}$$

whose solution is given by

$$\hat{\mathbf{q}}^{(j+1)} = \arg\max_{\mathbf{p}_q} \mathbf{v}_1(\mathbf{F}_{(q)}^{(j)}, \mathbf{G}_{(q)}^{(j)}), \tag{29}$$

and

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q} \lambda_1(\mathbf{F}_{(q)}^{(j)}, \mathbf{G}_{(q)}^{(j)}), \tag{30}$$

where

$$\mathbf{F}_{(q)}^{(j)} = \mathbf{L}^T(\mathbf{p}_q) \mathbf{Q}_{(q)}^{(j)} \mathbf{C} \mathbf{Q}_{(q)}^{(j)} \mathbf{L}(\mathbf{p}_q), \tag{31}$$

and

$$\mathbf{G}_{(q)}^{(j)} = \mathbf{L}^T(\mathbf{p}_q) \mathbf{Q}_{(q)}^{(j)} \mathbf{Q}_{(q)}^{(j)} \mathbf{L}(\mathbf{p}_q), \tag{32}$$

with $\mathbf{v}_1(\mathbf{F}, \mathbf{G})$ and $\lambda_1(\mathbf{F}, \mathbf{G})$ denoting the generalized eigenvector and generalized eigenvalue, respectively, corresponding to the maximum generalized eigenvalue of the matrix pencil $\mathbf{F}, \mathbf{G}$.

It is interesting to compare the above derived AP algorithms to the seemingly similar recursively applied and projected (RAP) beamformer [1], which is shown in [1] to be mathematically equivalent to the multiple constrained minimum variance (MCMV) [18]. To this end, we write down the RAP beamformer for a fixed-oriented dipole as

$$\hat{\mathbf{p}}_q = \arg\max_{\mathbf{p}_q} \frac{\mathbf{l}^T(\mathbf{p}_q) \mathbf{Q}_{(q)}^{(0)} \mathbf{l}(\mathbf{p}_q))}{\mathbf{l}^T(\mathbf{p}_q) (\mathbf{Q}_{(q)}^{(0)} \mathbf{C} \mathbf{Q}_{(q)}^{(0)})^{\dagger} \mathbf{l}(\mathbf{p}_q)}, \tag{33}$$

where $\dagger$ denotes the pseudo-inverse.

Comparing (33) to (22), or to (25) which is more inline with it, it is clear that the nominator of (33) is identical to the denominator of (25), while the denominator of (33), excluding the pseudoinverse, is identical to the nominator of (25). Note that the pseudoinverse "compensates" for this role switching of the nominator and the denominator, making the two algorithms seemingly similar. Computationally, (22) and (25) are much simpler than (33) since no psuedoinverse is required. Moreover, they work even for the case of a single sample and for the case of synchronous sources, i.e., when $\mathbf{C}$ is rank-1, in which case (33) all the LCMV-based beamformers fail. Another difference between the algorithms is their iterative nature. While (33) is computed only Q times, for $q = 1, ..., Q$, the AP algorithm (22) is computed iteratively till convergence, thus enabling improved performance.

Since the number of sources Q is assumed known, we can replace the covariance matrix $\mathbf{C}$ in (22) by its signal-subspace approximation given by $\mathbf{U}_s \mathbf{\Lambda}_s \mathbf{U}_s^T$, where $\mathbf{\Lambda}_s$ is the matrix of the Q largest eigenvalues of $\mathbf{C}$, and $\mathbf{U}_s$ is the matrix of the corresponding eigenvectors. We get:

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q} \frac{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(j)}\mathbf{U}_s\mathbf{\Lambda}_s\mathbf{U}_s^T\mathbf{Q}_{(q)}^{(j)}\mathbf{l}(\mathbf{p}_q))}{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(j)}\mathbf{l}(\mathbf{p}_q)}, \tag{34}$$

This algorithm can be regarded as the iterative and sequential version of the weighted-MUSIC algorithm [23], and we refer to it as AP-wMUSIC.

If we omit the weighting matrix $\mathbf{\Lambda}_s$ from this expression we get:

$$\hat{\mathbf{p}}_q^{(j+1)} = \arg\max_{\mathbf{p}_q} \frac{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(j)}\mathbf{U}_s\mathbf{U}_s^T\mathbf{Q}_{(q)}^{(j)}\mathbf{l}(\mathbf{p}_q))}{\mathbf{l}^T(\mathbf{p}_q)\mathbf{Q}_{(q)}^{(j)}\mathbf{l}(\mathbf{p}_q)}, \tag{35}$$

which is the same form as the sequential MUSIC (S-MUSIC) [24], with the difference being that (35) is computed iteratively till convergence while the S-MUSIC is computed only Q times for $q = 1, ...Q$, with no iterations. These iterations enable (35) to work well also for synchronous sources, unlike S-MUSIC. We refer to (35) as AP-MUSIC. We should point out that an algorithm similar to (35), referred to as self-consistent MUSIC has been introduced in [25]. Yet, being based on RAP-MUSIC [9], this algorithm is computationally more complex.

Once the AP algorithm has converged to $\hat{\mathbf{P}} = [\hat{\mathbf{p}}_1, ..., \hat{\mathbf{p}}_Q]$, it follows from (10) and (6) that the time-course of the individual signals are given by

$$\hat{\mathbf{S}} = [\hat{\mathbf{s}}_1^T, ..., \hat{\mathbf{s}}_Q^T]^T = (\mathbf{A}(\hat{\mathbf{P}})^T\mathbf{A}(\hat{\mathbf{P}}))^{-1}\mathbf{A}(\hat{\mathbf{P}})^T\mathbf{Y}. \tag{36}$$

IV. Concluding Remarks

We have presented a new sequential and iterative solution to the localization of EEG/MEG signals based on minimizing the LS criterion by the AP algorithm. This solution is applicable to a single sample and to the case of synchronous sources, and is computationally similar to all the sequential competing algorithms based on the LCMV and MUSIC frameworks, with the difference that the algorithm iterates till convergence. As a by product, we have also derived signal subspace versions of this algorithm, referred to as IS-weighted-MUSIC and IS-MUSIC which are applicable even in the case of synchronous sources.

V. Simulation Results

VI. Conclusions

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